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MATHEMATICAL MODEL OF FROZEN CONSUMPTION PRODUCTS

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Abstract—The paper deals with a mathematical model of the process of freezing consumption products. One-dimensional symmetrical problem with the third kind boundary conditions is considered. Differential equations for heat and mass transfer (for discontinuous and nonlinear coefficient) in accordance with the investigated process were numerically solved by finite difference method. The analysis of the results obtained is presented here.

NOMENCLATURE

a, coefficient of temperature balance;
c, c_1, c_2, specific heat;
$\bar{c}_i$, average integral specific heat ($i = 0, 1, 2$);
F, initial humidity distribution;
h, steps of length;
i, $(= 1 \ldots 2n)$; j, $(= 1, 2)$; l, $(= 1 \ldots v)$; n, $(= 1, 2 \ldots)$; z, $(= l, l+1)$ — integers;
k, steps of time;
k', k_1, k_2, coefficient of mass conduction;
r, $(= 1, 2 \ldots)$ integer;
L, body thickness;
p, constant;
T, temperature;
$\bar{T}_i$, average integral temperature ($i = 1, 2$);
T_{kr}, cryoscopic temperature for meat $T_{kr} = 272{\cdot}16°K$;
T_{k0}, finite temperature of meat at the plate centre;
T_0, initial temperature of meat;
T_n, surface temperature ($x = n$) in time τ;
T_s, surrounding medium temperature in time τ;
w, mass-transfer potential;
w_n, mass-transfer potential on surface ($x = n$) in time τ;
w_r, balanced mass-transfer potential;
w_{kr}, mass-transfer potential on surface ($T = T_{kr}$);
V, enthalpy of frozen products;
x, length coordinate;
y, distance of limit phase separation from the surface in time τ;
$\hat{y}$, approximate function for values of y.

Greek symbols

α, heat-transfer coefficient;
ε_n, different values;
ϑ, approximate function for values of T;
$\bar{\vartheta}$, average value;
κ, coefficient of mass transfer;
λ, thermal conductivity;
v, $(= 1, 2 \ldots)$ integer;
ρ, product density;
τ, time;
υ, displacement velocity of interface;
ω, approximate function for values of w.

1. INTRODUCTION

DETERMINATION of unsteady temperature field of frozen food products is a very difficult problem because of the complicated character of such a process. It differs from ordinary cooling by the following features:

(a) In examining the product it contains heat sources after ice formation;

(b) Physical parameters of a product (thermal conductivity λ, specific heat c, coefficient of temperature balance a) depend on temperature and possess cryoscopic temperature T_{kr} which is discontinuous and nondifferential (Fig. 1).

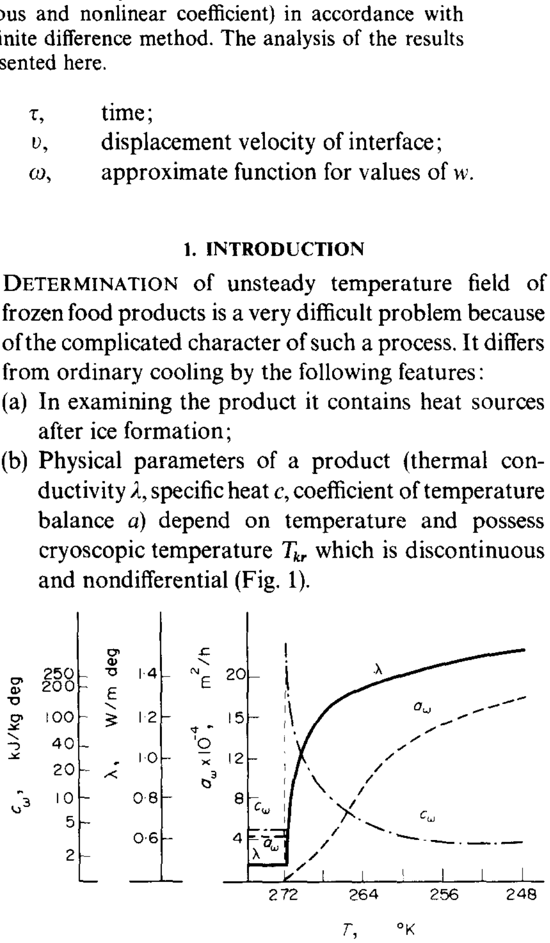

FIG. 1. Relation between heat-transfer coefficient of frozen meat and temperature.

Discontinuity and nonlinear physical parameters result in nonisothermal transformation of water phase into ice (solid in many water solutions).

Due to the complicated character of a process the theory of freezing food products contains many unsolved problems among which the most important are related to:

1. The lack of enough exact solution for determining the time of freezing of simple configuration products; disregard of the complicated configuration for which the problem is practically unsolved;

2. The lack of solution for unsteady temperatue field as well as for problems of crystal body shifting in case of frozen products of simple and complex configuration;
3. The lack of sufficient experimental data on physical parameters during freezing of food products with regard for moisture motion.

At present there is a feeling of lack of publications on solution (even approximate) of any of the above problems.

It is assumed in available publications [1–5] concerning freezing problems that at the interface there occurs freezing of the whole moisture (water) at a constant temperature, and physical parameters λ, c, a change their properties when $T \geqslant T_{kr}$. As far as non-isothermal process of freezing of water contained in food products is concerned, such an assumption is unacceptable. In [6] dealing with the determination of an unsteady temperature field and time of freezing of food products, phase separation which results in a very limited range was not taken into account.

In this work a mathematical model is given which realizes a freezing process at conditions close to reality. The above mathematical model made it possible to solve many problems which allow better understanding of the complicated process.

2. MATHEMATICAL MODEL

Due to the complicated character, the problem stated was confined to a one-dimensional case (unlimited plane).

Let the food products (meat, fish) be homogeneous isotropic materials whose physical parameters depend mainly on their temperature and moisture.

On both sides of the material the conditions of heat and mass transfer are identical (symmetrical problem).

At a discontinuity point there exists a separation limit phase, at which

$$T(y, \tau) = T_{kr} = \text{const.}$$

This limit divides the examined medium into unfrozen part 1 and frozen part 2 (Fig. 2).

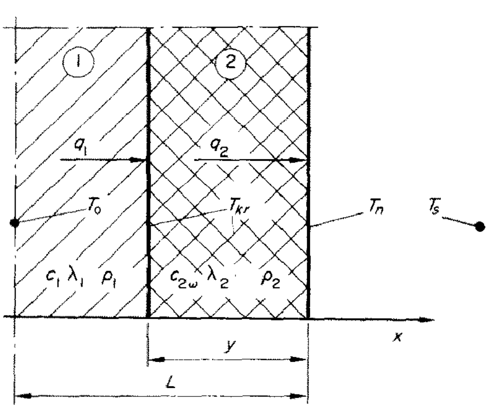

FIG. 2. Frozen meat plate.

From the heat balance we have

$$\frac{\partial V}{\partial \tau} = \alpha . (T_s - T_n)$$

$$\frac{\partial}{\partial \tau}[c_1(w) . \bar{T}_1 . (L-y) . \rho_1 + c_2(w, T) . \bar{T}_2 . y . \rho_2] = \alpha . (T_s - T_n). \quad (1)$$

Taking into consideration heat and mass transfer of freezing process, we may describe the process by the following equations

$$\rho_1 . c_1(w) \frac{\partial T}{\partial \tau} = \frac{\partial}{\partial x}\left[\lambda_1(w) . \frac{\partial T}{\partial x}\right] \quad 0 \leqslant x < L-y \quad (2)$$

$$\frac{\partial w}{\partial \tau} = \frac{\partial}{\partial x}\left[k_1(w) . \frac{\partial w}{\partial x}\right] \quad 0 \leqslant x < L-y \quad (3)$$

$$\rho_2 . c_2(w, T) \frac{\partial T}{\partial \tau} = \frac{\partial}{\partial x}\left[\lambda_2(w, T) \frac{\partial T}{\partial x}\right] \quad L-y < x < L \quad (4)$$

$$\frac{\partial w}{\partial \tau} = \frac{\partial}{\partial x}\left[k_2(w, T) . \frac{\partial w}{\partial x}\right] \quad L-y < x < L \quad (5)$$

$$\alpha . (T_n - T_s) = -\lambda(w, T) . \left.\frac{\partial T}{\partial x}\right|_{x=L} \quad (6)$$

$$\kappa(w_n - w_r) = -k'(w, T) \left.\frac{\partial w}{\partial x}\right|_{x=L} \quad (6')$$

$$\begin{aligned} &T(x, 0) = \psi(x), \ T(y, \tau) = T_{kr}, \\ &w(x, 0) = F(x), \ w(y, \tau) = w_{kr}, \ T(0, \tau_k) = T_{k0}. \end{aligned} \quad (7)$$

The system of equations (1)–(7) cannot be solved analytically. The detailed solution of such a system can be obtained only by a numerical method.

3. NUMERICAL SOLUTION OF THE MODEL PROPOSED

Of the known numerical methods only the network method [7, 8] allows rather exact solution for complicated nonlinear heat and mass-transfer problems.

The above assumptions confine the solution for food freezing problems to:

$$\bar{D}[0 \leqslant x \leqslant L, 0 \leqslant \tau \leqslant \tau_v].$$

Covering the range D with a difference net divides $0 \leqslant x \leqslant L$ into n equal segments with h as the length; we have

$$x_i = (i-1) . h, \ x_n = L, \ i = 1 \dots n.$$

Dividing the time interval $0 \leqslant \tau \leqslant \tau_v$ into v equal segments with k as length, obtain

$$\tau_l = l . k, \ l = 0 \dots v.$$

Operating through the divided points parallel and straight to the coordinates (τ, x), obtain the difference net $\bar{D}_{hk}$ covering the range $\bar{D}$ (Fig. 3).

The proposed mathematical model (1)–(7) could be approximated in the following difference analogy

$$\delta_\tau[\bar{c}_1 . \bar{\vartheta}_1 . (L-\hat{y}) . \rho_1 + \bar{c}_2 . \bar{\vartheta}_2 . \hat{y} . \rho_2] = \alpha . (T_{s,l+1} - \vartheta_{n,l+1}), \quad 0 < \hat{y} \leqslant L \quad (8)$$

$$(\rho_j c_j)_{i,l} \delta_\tau \vartheta_{i,l+1} = (\lambda_j)_{i,l} . (\delta_{xx} \vartheta_{i,l} + \delta_x(\lambda_j)_{i,l} . \delta_x \vartheta_{i,l} \quad (9)$$

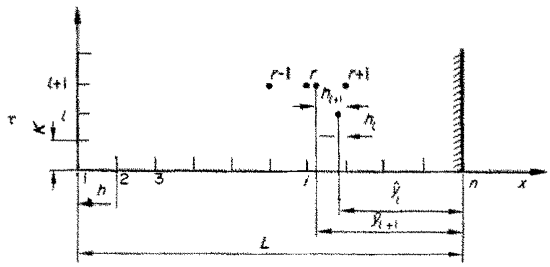

FIG. 3. Net difference.

$$\delta_\tau \omega_{i,l+1} = (k_j)_{i,l}\delta_{xx}\omega_{i,l}+\delta_x(k_j)_{i,l}\cdot\delta_x\omega_{i,l} \qquad (10)$$

$$\begin{array}{lll} \hat{y}=0 & i=1\ldots n-1, & j=1 \\ 0<\hat{y}<h & i=1\ldots n-2, & j=1 \\ h<\hat{y}<2h & i=1\ldots n-3, & j=1 \\ 2h<\hat{y}<L & i=1\ldots r-1, & j=1 \\ & i=r+2\ldots n-1, & j=2 \end{array}$$

$$(\rho_j c_j)_{i,l+1}\,\delta_\tau\vartheta_{i,l+1} = (\lambda_j)_{i,l+1}\,\delta_{xx}\vartheta_{i,l+1}+\delta_x(\lambda_j)_{i,l+1}\,\delta_x\vartheta_{i,l+1} \qquad (11)$$

$$\delta_\tau\omega_{i,l+1} = (k_j)_{i,l+1}\,\delta_{xx}\omega_{i,l+1}+\delta_x(k_j)_{i,l+1}\,\delta_x\omega_{i,l+1} \qquad (12)$$

$$\begin{array}{lll} 0<\hat{y}<h & i=r, & j=1 \\ h<\hat{y}<L-h & i=r, & j=1 \\ & i=r+1, & j=2 \\ \vartheta(0,\tau)<\vartheta_{kr} & i=1\ldots n-1, & j=2 \end{array}$$

$$\alpha(\vartheta_{n,l+1}-T_{s,l+1}) = -\lambda_{n,l+1}\,\delta_x\vartheta_{n,l+1} \qquad (13)$$

$$\kappa(\omega_{n,l+1}-Wr_{l+1}) = -k'_{n,l+1}\,\delta_x\omega_{n,l+1} \qquad (13')$$

$$\begin{array}{l} \vartheta_{i,0}=\psi'(x), \quad \vartheta(\hat{y},\tau)=T_{kr}, \quad \omega_{i,0}=F'(x), \\ \omega(\hat{y},\tau)=w_{kr}, \quad \vartheta(0,\tau_v)=T_{k0} \end{array} \qquad (14)$$

Equations (8)–(14) are solved using difference diagrams in the form I and II (Fig. 4).

Difference diagram I was used in the case when the separation limit existed.

When the separation limit disappeared, the diagram II was used. At the initial time instant ϑ_{ip}, $\omega_{i,0}$, $\hat{y}_0$ are given. In order to calculate these values at the time instant $l+1$ equations (9), (10) and (13) were used. When $\vartheta_{n,l}<\vartheta_{kr}$ equations (8)–(12) and (13) were used, while at $\vartheta_{1,l}<\vartheta_{kr}$ equations (11)–(13) were employed.

4. STABLE AND CONVERGENT DIFFERENCE DIAGRAM

The problem of stable and convergent solution of differential equations (for nonlinear heat-conduction problems) are very little investigated at present.

The only reliable way is to compare the results obtained for different values of h and k.

The stability problem is related to the main difference diagram (diagrams II and absolutely stable).

The stable solution (for constant coefficients) results in a definite time k with respect to:

$$k \leqslant \frac{p\,.\,h^2}{(a_{i,l})_{\max}} \qquad (15)$$

where p is constant.

Large values of h and k lead to considerable errors while small values cause sufficient increase in calculation time. Then it is necessary to find an optimal value of h and k to ensure the exactness requirement with a minimum calculation time, k being calculated from equation (15), calculation is repeated with the value of $2h$. If the difference in calculation results is small, the

where

$$\delta_\tau\beta_{i,l+1} = \frac{\beta_{i,l+1}-\beta_{i,l}}{k} \qquad \beta=\vartheta,\omega$$

$$\delta_x\beta_{i,z} = \frac{\beta_{i+1,z}-\beta_{i-1,z}}{2h} \qquad \beta=\lambda,\vartheta,\omega$$

$$\delta_{xx}\beta_{i,z} = \frac{\beta_{i+1,z}-2\beta_{i,z}+\beta_{i-1,z}}{h^2} \qquad \beta=\vartheta,\omega$$

$$\begin{array}{llllll} \hat{y}=0 & i=1\ldots n-1, & z=l; & 0<\hat{y}<h & i=1\ldots n-2, & z=l \\ h<y\leqslant 2h & i=1\ldots n-3, & z=l; & 2h<\hat{y}\leqslant L & i=r-1,\,r+2\ldots n-1, & z=l \\ \vartheta_{0,L}<T_{kr} & i=1\ldots n-1, & z=l+1; & & & \end{array}$$

$$\delta_x\beta_{i,z} = \frac{\beta_{i+1,z}-\beta_{kr}}{h+h_z} \qquad \beta=\lambda,\vartheta,\omega$$

$$\delta_{xx}\beta_{i,z} = \frac{2}{h+h_z}\left[\frac{\beta_{i+1,z}-\beta_{i,z}}{h}-\frac{\beta_{i,z}-\beta_{kr}}{h_z}\right] \qquad \beta=\vartheta,\omega$$

$$h\leqslant\hat{y}<L \quad i=r+1, \quad z=l, \quad z=l+1$$

$$\delta_x\beta_{i,z} = \frac{\beta_{kr}-\beta_{i-1,z}}{2h-h_z} \qquad \beta=\lambda,\vartheta,\omega$$

$$\delta_{xx}\beta_{i,z} = \frac{2}{2h-h_z}\left[\frac{\beta_{kr}-\beta_{i,z}}{h-h_z}-\frac{\beta_{i,z}-\beta_{i-1,z}}{h}\right] \qquad \beta=\vartheta,\omega$$

$$0<\hat{y}<L-h \quad i=r, \quad z=l, \quad z=l+1$$

$$\delta_x\beta_{n,l+1} = \frac{\beta_{n,l+1}-\beta_x}{\zeta} \qquad \beta=\vartheta,\omega$$

$$\hat{y}=0, \quad \hat{y}\geqslant h \qquad \beta=\vartheta_{n-1,l+1};\ \zeta=h$$

$$\beta_x=\omega_{n-1,l+1};\ \zeta=h$$

$$0<\hat{y}<h. \qquad \beta_x=\vartheta_{kr},\omega_{kr};\ \zeta=h_{l+1}$$

$$\beta=\vartheta,\omega$$

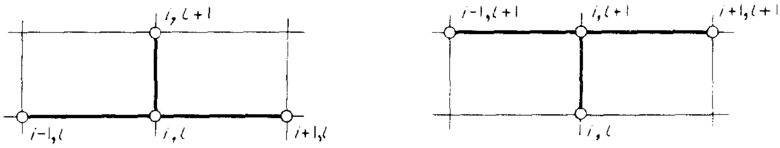

FIG. 4. Diagrams I and II.

value of h is not greater than the optimal one. In the contrary case the value of $h/2$ should be used. If the difference in calculation results is not small, it means that the value of h is either greater or less than the optimal one.

The value of p is chosen when calculating with different values $(0 < p \leqslant \frac{1}{2})$ and shows, whether the solution is convergent or divergent.

5. CALCULATION EXAMPLE

The universal difference model permits freezing parameter influence (α, T_s, T_p, δ) on the time of the process, final temperature difference ($\Delta T = T_{1,k} - T_{n,k}$), displacement speed of interface ($v = \mathrm{d}\hat{y}/\mathrm{d}\tau$) to be analysed.

Calculations are carried out only for beef meat, whose heat-conduction coefficients a, λ are expressed by the following relations:

for $248{\cdot}16 \leqslant T \leqslant T_{kr}$

$$a(T) = [-11032{\cdot}6509 - 261{\cdot}421965 \times Z - 18252{\cdot}4705 \times Z^2 - 26{\cdot}133707 \times Z^3 - 133{\cdot}87762 \times Z^4 + - 7{\cdot}31961337 \times Z^5 - 0{\cdot}11645322 \times Z^6] \times 10^{-9},\ \mathrm{m^2/h}$$

$$\lambda(T) = [-124634{\cdot}1825 - 744465{\cdot}0959 \times Z - 160251{\cdot}468 \times Z^2 - 17695{\cdot}612 \times Z^3 - 102{\cdot}877816 \times Z^4 + 29{\cdot}874844 \times Z^5 - 0{\cdot}343013138 \times Z^6] \times 10^{-6},\ \mathrm{W/m\,.deg}$$

for $T_{kr} \leqslant T \leqslant 303{\cdot}16^\circ\mathrm{K}$

$$a(T) = 0{\cdot}00042 - 0{\cdot}000001 \times z,\ \mathrm{m^2/h}$$

$$\lambda(T) = 0{\cdot}476079324 - 0{\cdot}0004026324 \times z,\ \mathrm{W/m\,.deg}$$

where $z = T - 273{\cdot}16$.

Due to the lack of experimental data the influence of the moisture motion on a temperature distribution was neglected in the calculation.

Figures 5 and 6 show typical functions $T = T(x, \tau)$ and $\hat{y} = \hat{y}(\tau)$ for the following data:

$L = 0{\cdot}08\,\mathrm{m}$	$h = 0{\cdot}01\,\mathrm{m}$	$k = 0{\cdot}04\,\mathrm{h}$
$T_s = 257{\cdot}16^\circ\mathrm{K}$	$T_0 = 279{\cdot}16^\circ\mathrm{K}$	$T_{k0} = 260{\cdot}66^\circ\mathrm{K}$
$\alpha_s = 33{\cdot}26\,\mathrm{W/m^2.deg}$	$\rho_1 = 1070\,\mathrm{kg/m^3}$	$\rho_2 = 1010\,\mathrm{kg/m^3}$.

Calculation error ε_n resulted from the replacement of the differential equation by finite-difference one solved by the Runge method is verified by the comparison of the calculated results with different values of the net range.

$$\varepsilon_n \approx 0{\cdot}2^\circ\mathrm{K}.$$

Here presented is the effect of freezing parameters on the time of process (Figs. 7 and 8), final difference temperature (Fig. 9), displacement interface (Fig. 10).

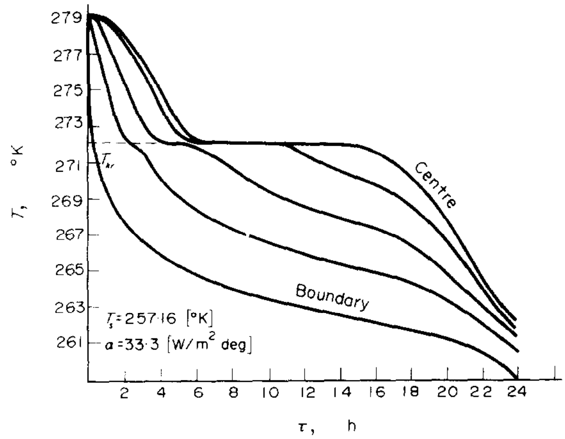

FIG. 5. Change of temperature of frozen meat plate.

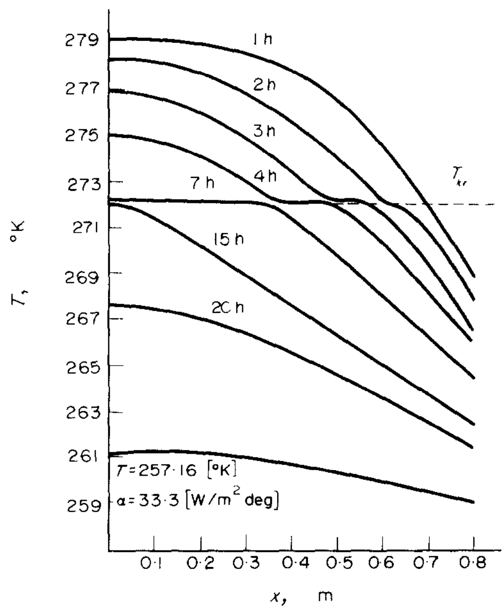

FIG. 6. Temperature distribution along the thickness of mean plate.

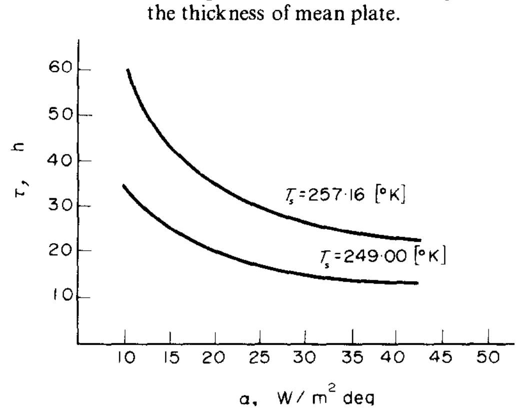

FIG. 7. Influence of heat-transfer coefficient on freezing time.

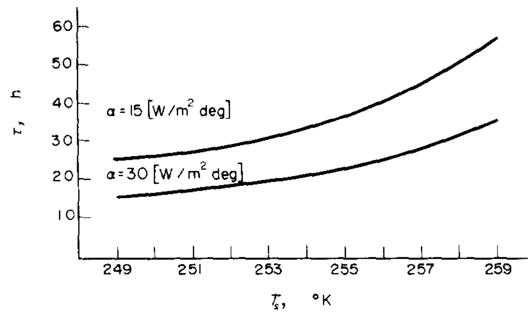

FIG. 8. Influence of surrounding cooling temperature on time duration of freezing process.

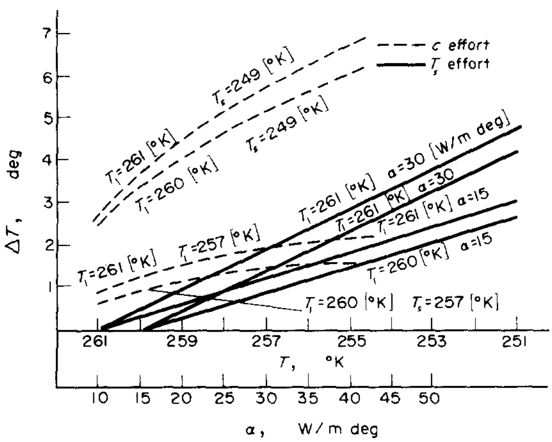

FIG. 9. Influence of heat-transfer coefficient and surrounding cooling temperature on product final difference in temperature.

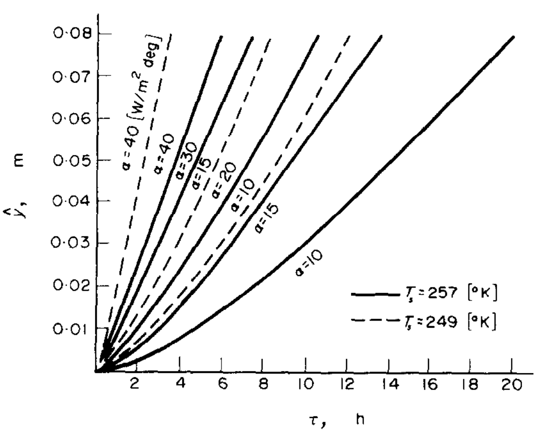

FIG. 10. Influence of heat-transfer coefficient and surrounding cooling temperature on interphase transfer.

6. CONCLUSIONS

From the analysis of the temperature variation diagram (Fig. 5) two different stages in a process of freezing can be seen. At the first stage a decrease in temperature (to T_{kr}) is seen in all layers of the first sample, being the quickest on the surface and slowest in the middle.

The fall of temperature is retarded due to the effect of the solidified water heat, which is particularly large near the cryoscopic temperature (T_{kr}). The more intensive cooling of the test layers, the more pronounced is the temperature decrease when passing through T_{kr}.

In case of slow cooling the retardation of temperature fall is distinct (particularly at the centre of the sample).

Rapid temperature fall whose effect is more clearly observed at the centre of the sample is also seen in the figure. This is again the effect of solidified water solution with noneutectic components. At this stage the basic part of the heat removed is used for freezing out water in the central part of the sample, while in all other layers most of the water is already frozen out. When most of the water at the centre of the sample is frozen out, an accelerated fall of temperature is observed on its surface. It can be seen from Fig. 6 that for steady conditions (α = const, τ_s = const) the velocity of interface is approximately constant. This velocity increases with the rate of heat removal.

From the analysis of the calculation carried out the following conclusions of practical value can be made:

1. The thickness of meat should not be large to give the most effective time of freezing and a small final temperature difference ΔT.
2. The rate of heat removal should be the highest at the onset of the process. It provides short freezing time τ, low final temperature difference ΔT and high velocity of interface displacement.
3. Initial meat temperature ($T_p > T_{kr}$) has a small effect on τ and ΔT. For $T_p < T_{kr}$, T_p is observed to affect the time of process.
4. At final temperature difference ΔT the greatest effect is exerted by the cooling temperature T_s.
5. Increase of the cooling rate involves decrease of τ and increase of ΔT.
6. The relationship between the cooling rate and the time of process is nonlinear.

The comparison of the analytical and experimental data proved the feasibility of the model proposed.

The maximum error in numerical determination of the process time does not exceed 5 per cent compared to experimental data [9]. Temperature history diagram $T(x, \tau)$ plotted in accordance with the experimental data [9] shows a clear retardation of temperature decrease when passing through T_{kr}.

This can be attributed to large measurement time (2 h), unexact measurement of temperature and other physical and chemical characteristics of the test sample. The characteristics of temperature variation in both (experimental and numerical) cases are very similar.

Due to very complicated practical investigation (the method presented is in the mean time the only one which allows an exact analysis of the process of food staff freezing.

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MODELE MATHEMATIQUE DE CONGELATION DES PRODUITS DE CONSOMMATION

Résumé—L'article traite d'un modèle mathématique du processus de congélation des produits de consommation. On considère un problème symétrique unidimensionnel avec conditions aux limite de 3ème espèce. Les équations différentielles du transfert de chaleur et de masse (avec coefficients discontinus et non linéaires) relatives au processus étudié sont résolues numériquement à l'aide d'une méthode de différences finies. L'analyse des résultats obtenus est présentée dans l'article.

MATHEMATISCHES MODELL FÜR GEFRIERPRODUKTE

Zusammenfassung—Die Arbeit befaßt sich mit einem mathematischen Modell für den Ablauf des Gefrierens in Nahrungsmitteln. Es wird das eindimensionale symmetrische Problem mit der Randbedingung 3. Art betrachtet. Für den untersuchten Fall werden die Differentialgleichungen für den Wärme- und Stoffaustausch (für diskontinuierliche und nichtlineare Koeffizienten) mit Hilfe der Methode finiter Differenzen numerisch gelöst. Die gewonnenen Ergebnisse werden diskutiert.

МАТЕМАТИЧЕСКАЯ МОДЕЛЬ ЗАМОРОЖЕННЫХ ПРОДУКТОВ ПОТРЕБЛЕНИЯ

Аннотация — Рассматривается математическая модель процесса замораживания продуктов потребления. Обсуждается одномерная симметричная задача с граничными условиями третьего рода. С помощью метода конечных разностей численно решаются дифференциальные уравнения тепло- и массообмена (при разрывном и нелинейном коэффициенте). Представлен анализ полученных результатов.